Chapter 6: Nonignorable Missing Data

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Introduction

- (X, Y): random variable, y is subject to missingness
- Response indicator function

$$\delta_i = \begin{cases} 1 & \text{if } y_i \text{ is observed} \\ 0 & \text{otherwise.} \end{cases}$$

- Nonignorable nonresponse

$$f(y \mid \mathbf{x}) \neq f(y \mid \mathbf{x}, \delta = 1).$$

- In general,

$$f(y \mid \mathbf{x}, \delta = 1) = \frac{P(\delta = 1 \mid \mathbf{x}, y)}{P(\delta = 1 \mid \mathbf{x})} f(y \mid \mathbf{x}).$$

  Thus, $P(\delta = 1 \mid \mathbf{x}, y) \neq P(\delta = 1 \mid \mathbf{x})$ implies nonignorable nonresponse.

Observed likelihood

- $f(y \mid \mathbf{x}; \theta)$: model of y on $\mathbf{x}$
- $g(\delta \mid \mathbf{x}, y; \phi)$: model of δ on $(\mathbf{x}, y)$
- Observed likelihood

$$
\begin{aligned}
L_{obs}(\theta, \phi) &= \prod_{\delta_i = 1} f\left(y_i \mid \mathbf{x}_i; \theta\right) g\left(\delta_i \mid \mathbf{x}_i, y_i; \phi\right) \\
&\quad \times \prod_{\delta_i = 0} \int f\left(y_i \mid \mathbf{x}_i; \theta\right) g\left(\delta_i \mid \mathbf{x}_i, y_i; \phi\right) dy_i
\end{aligned}
$$

- Under what conditions are the parameters identifiable (or estimable)?

Definition

Identifiability

Let $\mathcal{P} = \{P_\theta; \theta \in \Theta\}$ be a statistical model with parameter space in Θ. We say that $\mathcal{P}$ is identifiable if the mapping $\theta \to P_\theta$ is one-to-one:

$$P_{\theta_1} = P_{\theta_2} \;\text{ implies }\; \theta_1 = \theta_2 \;\text{ for all } \theta_1, \theta_2 \in \Theta.$$

That is, if $F(\mathbf{z}; \theta)$ is the distribution function from P_θ then for any θ_1 and θ_2 in Θ such that $\theta_1 \neq \theta_2$, it implies

$$F(\mathbf{z}; \theta_1) \neq F(\mathbf{z}, \theta_2)$$

for some $\mathbf{z}$.

Remark

Identifiability is a concept closely related to the ability to estimate the parameters of a model from a sample generated by the model.

Example 1

Measurement error models

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 x_i + e_i \\ X_i &= x_i + u_i \end{aligned}$$

where $(x_i, e_i, u_i)' \sim N[(\mu_x, 0, 0), \mathrm{diag}(\sigma_{xx}, \sigma_{ee}, \sigma_{uu})]$. We observe (X_i, Y_i) from the sample. In this case, we have

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_x \\ \beta_0 + \beta_1 \mu_x \end{pmatrix}, \begin{pmatrix} \sigma_{xx} + \sigma_{uu} & \beta_1 \sigma_{xx} \\ \beta_1 \sigma_{xx} & \sigma_{ee} + \beta_1^2 \sigma_{xx} \end{pmatrix}\right].$$

The joint distribution is completely determined by five sufficient statistics and is a function of six parameters. Thus, the distribution is not identified.

Example 2

- x, y: dichotomous (taking 0 or 1).
- x is always observed and y is subject to missingness
- Response model

$$P(\delta = 1 \mid x, y) = \frac{\exp(\phi_0 + \phi_1 x + \phi_2 y + \phi_3 xy)}{1 + \exp(\phi_0 + \phi_1 x + \phi_2 y + \phi_3 xy)}$$

- The model is not identified because the number of sufficient statistics is smaller than the number of parameters.
- If the response mechanism satisfies $P(\delta = 1 \mid x, y) = P(\delta = 1 \mid y)$, then the model is identified.

Example 3

- x, z, y: dichotomous (taking 0 or 1).
- (x, z) is always observed and y is subject to missingness
- If the response mechanism satisfies
  $P(\delta = 1 \mid x, z, y) = P(\delta = 1 \mid x, y)$, then the model is identified.

Lemma (Wang et al., 2014)

Suppose that we can decompose the covariate vector $\mathbf{x} = (\mathbf{u}, \mathbf{z})$ such that

$$g(\delta|y, \mathbf{x}) = g(\delta|y, \mathbf{u}) \tag{1}$$

and, for any given $\mathbf{u}$, there exist $z_{\mathbf{u},1}$ and $z_{\mathbf{u},2}$ such that

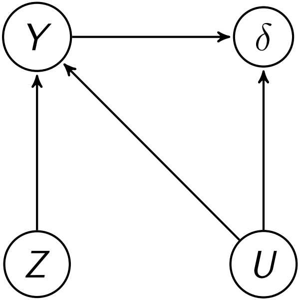
$$f(y|\mathbf{u}, \mathbf{z} = z_{\mathbf{u},1}) \neq f(y|\mathbf{u}, \mathbf{z} = z_{\mathbf{u},2}). \tag{2}$$

Under some other minor conditions, all the parameters in f and g are identifiable.

Remark

- Condition (1) means
$$\delta \perp \mathbf{z} \mid y, \mathbf{u}.$$
- That is, given $(y, \mathbf{u})$, $\mathbf{z}$ does not help in explaining δ.

Figure: A DAG for understanding nonresponse instrumental variable Z



- We may call $\mathbf{z}$ the nonresponse instrument variable.

§1. Full likelihood-based ML estimation

Full likelihood-based ML estimation

- Wish to find $\hat{\eta} = (\hat{\theta}, \hat{\phi})$, that maximizes the observed likelihood

$$
\begin{aligned}
L_{obs}(\eta) &= \prod_{\delta_i = 1} f(y_i \mid \mathbf{x}_i; \theta)\, g(\delta_i \mid \mathbf{x}_i, y_i; \phi) \\
&\quad \times \prod_{\delta_i = 0} \int f(y_i \mid \mathbf{x}_i; \theta)\, g(\delta_i \mid \mathbf{x}_i, y_i; \phi)\, dy_i
\end{aligned}
$$

- Mean score theorem: Under some regularity conditions, finding the MLE by maximizing the observed likelihood is equivalent to finding the solution to

$$
\bar{S}(\eta) \equiv E\{S(\eta) \mid \mathbf{y}_{obs}, \boldsymbol{\delta}; \eta\} = 0,
$$

where $\mathbf{y}_{obs}$ is the observed data. The conditional expectation of the score function is called mean score function.

EM algorithm

- Interested in finding $\hat{\eta}$ that maximizes $L_{obs}(\eta)$. The MLE can be obtained by solving $S_{obs}(\eta) = 0$, which is equivalent to solving $\bar{S}(\eta) = 0$ by the mean score theorem.
- EM algorithm provides an alternative method of solving $\bar{S}(\eta) = 0$ by writing

$$\bar{S}(\eta) = E\left\{S(\eta) \mid \mathbf{y}_{obs}, \boldsymbol{\delta}; \eta\right\}$$

and using the following iterative method:

$$\hat{\eta}^{(t+1)} \leftarrow \text{ solve } E\left\{S(\eta) \mid \mathbf{y}_{obs}, \boldsymbol{\delta}; \hat{\eta}^{(t)}\right\} = 0.$$

EM algorithm

Definition

Let $\eta^{(t)}$ be the current value of the parameter estimate of η. The EM algorithm can be defined as iteratively carrying out the following E-step and M-steps:

- E-step: Compute

$$Q\left(\eta \mid \eta^{(t)}\right) = E\left\{\ln f\left(\mathbf{y}, \boldsymbol{\delta}; \eta\right) \mid \mathbf{y}_{\mathrm{obs}}, \boldsymbol{\delta}, \eta^{(t)}\right\}$$

- M-step: Find $\eta^{(t+1)}$ that maximizes $Q(\eta \mid \eta^{(t)})$ w.r.t. η.

Monte Carlo EM

Motivation: Monte Carlo samples in the EM algorithm can be used as imputed values.

Monte Carlo EM

1. In the EM algorithm defined by
   - [E-step] Compute
   $$Q\left(\eta \mid \eta^{(t)}\right) = E\left\{\ln f\left(\mathbf{y}, \boldsymbol{\delta}; \eta\right) \mid \mathbf{y}_{\mathrm{obs}}, \boldsymbol{\delta}; \eta^{(t)}\right\}$$
   - [M-step] Find $\eta^{(t+1)}$ that maximizes $Q\left(\eta \mid \eta^{(t)}\right)$,

   E-step is computationally cumbersome because it involves integral.
2. Wei and Tanner (1990): In the E-step, first draw
   $$\mathbf{y}_{mis}^{*(1)}, \cdots, \mathbf{y}_{mis}^{*(m)} \sim f\left(\mathbf{y}_{mis} \mid \mathbf{y}_{\mathrm{obs}}, \boldsymbol{\delta}; \eta^{(t)}\right)$$
   and approximate

Remark

- Identifiability condition is needed to guarantee the convergence of EM sequence.
- The fully parametric model approach is known to be sensitive to the failure of model assumptions: Little (1985), Kenward and Molenberghs (1988)
- Sensitivity analysis is often recommended: Scharfstein et al. (1999)

§2. Partial Likelihood approach

Partial Likelihood approach

- A classical likelihood-based approach for parameter estimation under non ignorable nonresponse is to maximize $L_{obs}(\theta, \phi)$ with respect to (θ, ϕ), where

$$
\begin{aligned}
L_{obs}(\theta, \phi) &= \prod_{\delta_i=1} f\left(y_i \mid \mathbf{x}_i; \theta\right) g\left(\delta_i \mid \mathbf{x}_i, y_i; \phi\right) \\
&\quad \times \prod_{\delta_i=0} \int f\left(y_i \mid \mathbf{x}_i; \theta\right) g\left(\delta_i \mid \mathbf{x}_i, y_i; \phi\right) dy_i
\end{aligned}
$$

- Such approach can be called full likelihood-based approach because it uses full information available in the observed data.
- On the other hand, partial likelihood-based approach (or conditional likelihood approach) uses a subset of the sample.

Conditional Likelihood approach

Idea

- Since
$$f(y \mid \mathbf{x})g(\delta \mid \mathbf{x}, y) = f_1(y \mid \mathbf{x}, \delta)g_1(\delta \mid \mathbf{x}),$$
for some f_1 and g_1, we can write
$$\begin{aligned} L_{obs}(\theta) &= \prod_{\delta_i=1} f_1\left(y_i \mid \mathbf{x}_i, \delta_i = 1\right) g_1\left(\delta_i \mid \mathbf{x}_i\right) \\ &\quad \times \prod_{\delta_i=0} \int f_1\left(y_i \mid \mathbf{x}_i, \delta_i = 0\right) g_1\left(\delta_i \mid \mathbf{x}_i\right) dy_i \\ &= \prod_{\delta_i=1} f_1\left(y_i \mid \mathbf{x}_i, \delta_i = 1\right) \times \prod_{i=1}^{n} g_1\left(\delta_i \mid \mathbf{x}_i\right). \end{aligned}$$
- The conditional likelihood is defined to be the first component:
$$L_c(\theta) = \prod_{\delta_i=1} f_1\left(y_i \mid \mathbf{x}_i, \delta_i = 1\right) = \prod_{\delta_i=1} \frac{f(y_i \mid \mathbf{x}_i; \theta)\pi(\mathbf{x}_i, y_i)}{\int f(y \mid \mathbf{x}_i; \theta)\pi(\mathbf{x}_i, y)dy},$$
where $\pi(\mathbf{x}, y_i) = Pr(\delta_i = 1 \mid x_i, y_i)$.

Conditional Likelihood approach

Example

- Assume that the original sample is a random sample from an exponential distribution with mean $\mu = 1/\theta$. That is, the probability density function of y is $f(y; \theta) = \theta \exp(-\theta y) I(y > 0)$.
- Suppose that we observe y_i only when $y_i > K$ for a known $K > 0$.
- Thus, the response indicator function is defined by $\delta_i = 1$ if $y_i > K$ and $\delta_i = 0$ otherwise.

Conditional Likelihood approach

Example

- To compute the maximum likelihood estimator from the observed likelihood, note that

$$S_{\text{obs}}(\theta) = \sum_{\delta_i=1} \left(\frac{1}{\theta} - y_i \right) + \sum_{\delta_i=0} \left\{ \frac{1}{\theta} - E(y_i \mid \delta_i = 0) \right\}.$$

- Since

$$E(Y \mid y > K) = \frac{1}{\theta} - \frac{K \exp(-\theta K)}{1 - \exp(-\theta K)},$$

  the maximum likelihood estimator of θ can be obtained by the following iteration equation:

$$\left\{ \hat{\theta}^{(t+1)} \right\}^{-1} = \bar{y}_r - \frac{n-r}{r} \left\{ \frac{K \exp(-K\hat{\theta}^{(t)})}{1 - \exp(-K\hat{\theta}^{(t)})} \right\}, \quad (3)$$

  where $r = \sum_{i=1}^{n} \delta_i$ and $\bar{y}_r = r^{-1} \sum_{i=1}^{n} \delta_i y_i$.

Conditional Likelihood approach

Example

- Since $\pi_i = Pr(\delta_i = 1 \mid y_i) = I(y_i > K)$ and $E(\pi_i) = E\{I(y_i > K)\} = \exp(-K\theta)$, the conditional likelihood reduces to
$$\prod_{\delta_i=1} \theta \exp\{-\theta(y_i - K)\}.$$
The maximum conditional likelihood estimator of θ is
$$\hat{\theta}_c = \frac{1}{\bar{y}_r - K}.$$
Since $E(y \mid y > K) = \mu + K$, the maximum conditional likelihood estimator of μ, which is $\hat{\mu}_c = 1/\hat{\theta}_c$, is unbiased for μ.

Conditional Likelihood approach

Remark

- Under some regularity conditions, the solution $\hat{\theta}_c$ that maximizes $L_c(\theta)$ satisfies
$$\mathcal{I}_c^{1/2}(\hat{\theta}_c - \theta) \xrightarrow{\mathcal{L}} N(0, I)$$
where
$$\mathcal{I}_c(\theta) = -E\left\{\frac{\partial}{\partial\theta'} S_c(\theta) \mid \mathbf{x}_i; \theta\right\}$$
$S_c(\theta) = \partial \ln L_c(\theta)/\partial\theta$, and $S_i(\theta) = \partial \ln f\left(y_i \mid \mathbf{x}_i; \theta\right)/\partial\theta$.
- Works only when $\pi(x, y)$ is a known function.
- Does not require nonresponse instrumental variable assumption.
- Popular for biased sampling problem.

Pseudo Likelihood approach

Idea

- Consider bivariate (x_i, y_i) with density $f(y \mid x; \theta)h(x)$ where y_i are subject to missingness.
- We are interested in estimating θ.
- Suppose that $Pr(\delta = 1 \mid x, y)$ depends only on y. (i.e. x is nonresponse instrument)
- Note that $f(x \mid y, \delta) = f(x \mid y)$.
- Thus, we can consider the following conditional likelihood

$$L_c(\theta) = \prod_{\delta_i=1} f(x_i \mid y_i, \delta_i = 1) = \prod_{\delta_i=1} f(x_i \mid y_i).$$

- We can consider maximizing the pseudo likelihood

$$L_p(\theta) = \prod_{\delta_i=1} \frac{f(y_i \mid x_i; \theta)\hat{h}(x_i)}{\int f(y_i \mid x; \theta)\hat{h}(x)dx},$$

where $\hat{h}(x)$ is a consistent estimator of the marginal density of x.

Pseudo Likelihood approach

Idea

- We may use the empirical density in $\hat{h}(x)$. That is, $\hat{h}(x) = 1/n$ if $x = x_i$. In this case,

$$L_c(\theta) = \prod_{\delta_i=1} \frac{f(y_i \mid x_i; \theta)}{\sum_{k=1}^{n} f(y_i \mid x_k; \theta)}.$$

- We can extend the idea to the case of $\mathbf{x} = (\mathbf{u}, \mathbf{z})$ where $\mathbf{z}$ is a nonresponse instrument. In this case, the conditional likelihood becomes

$$\prod_{i:\delta_i=1} p(\mathbf{z}_i \mid y_i, \mathbf{u}_i) = \prod_{i:\delta_i=1} \frac{f(y_i \mid \mathbf{u}_i, \mathbf{z}_i; \theta) p(\mathbf{z}_i | \mathbf{u}_i)}{\int f(y_i \mid \mathbf{u}_i, \mathbf{z}; \theta) p(\mathbf{z} | \mathbf{u}_i) d\mathbf{z}}. \tag{4}$$

Pseudo Likelihood approach

- Let $\hat{p}(\mathbf{z}|\mathbf{u})$ be an estimated conditional probability density of $\mathbf{z}$ given $\mathbf{u}$. Substituting this estimate into the likelihood in (4), we obtain the following pseudo likelihood:

$$\prod_{i:\delta_i=1} \frac{f(y_i \mid \mathbf{u}_i, \mathbf{z}_i; \theta)\hat{p}(\mathbf{z}_i|\mathbf{u}_i)}{\int f(y_i \mid \mathbf{u}_i, \mathbf{z}; \theta)\hat{p}(\mathbf{z}|\mathbf{u}_i)dz}. \tag{5}$$

- The pseudo maximum likelihood estimator (PMLE) of θ, denoted by $\hat{\theta}_p$, can be obtained by solving

$$S_p(\theta;\hat{\alpha}) \equiv \sum_{\delta_i=1} [S(\theta;\mathbf{x}_i, y_i) - E\{S(\theta;\mathbf{u}_i,\mathbf{z},y_i) \mid y_i, \mathbf{u}_i;\theta,\hat{\alpha}\}] = 0$$

for θ, where $S(\theta;\mathbf{x},y) = S(\theta;\mathbf{u},\mathbf{z},y) = \partial \log f(y \mid \mathbf{x};\theta)/\partial\theta$ and

$$E\{S(\theta;\mathbf{u}_i,\mathbf{z},y_i) \mid y_i,\mathbf{u}_i;\theta,\hat{\alpha}\} = \frac{\int S(\theta;\mathbf{u}_i,\mathbf{z},y_i)f(y_i \mid \mathbf{u}_i,\mathbf{z};\theta)p(\mathbf{z} \mid \mathbf{u}_i;\hat{\alpha})dz}{\int f(y_i \mid \mathbf{u}_i,\mathbf{z};\theta)p(\mathbf{z} \mid \mathbf{u}_i;\hat{\alpha})dz}.$$

Pseudo Likelihood approach

- The Fisher-scoring method for obtaining the PMLE is given by

$$\hat{\theta}_p^{(t+1)} = \hat{\theta}_p^{(t)} + \left\{ \mathcal{I}_p \left(\hat{\theta}^{(t)}, \hat{\alpha} \right) \right\}^{-1} S_p(\hat{\theta}^{(t)}, \hat{\alpha})$$

where

$$\mathcal{I}_p(\theta, \hat{\alpha}) = \sum_{\delta_i = 1} \left[E\{S(\theta; \mathbf{u}_i, \mathbf{z}, y_i)^{\otimes 2} \mid y_i, \mathbf{u}_i; \theta, \hat{\alpha}\} - E\{S(\theta; \mathbf{u}_i, \mathbf{z}, y_i) \mid y_i, \mathbf{u}_i; \theta, \hat{\alpha}\}\right.$$

- First considered by Tang et al. (2003) and further developed by Zhao and Shao (2015).

§3. GMM approach

Basic setup

- (X, Y): random variable
- θ: Defined by solving

$$E\{U(\theta; X, Y)\} = 0.$$

- y_i is subject to missingness

$$\delta_i = \begin{cases} 1 & \text{if } y_i \text{ responds} \\ 0 & \text{if } y_i \text{ is missing.} \end{cases}$$

- Want to find w_i such that the solution $\hat{\theta}_w$ to

$$\sum_{i=1}^{n} \delta_i w_i U(\theta; x_i, y_i) = 0$$

is consistent for θ.

Basic Setup

- Result 1: The choice of

$$w_i = \frac{1}{E(\delta_i \mid x_i, y_i)} \tag{6}$$

makes the resulting estimator $\hat{\theta}_w$ consistent.

- Result 2: If $\delta_i \sim \text{Bernoulli}(\pi_i)$, then using $w_i = 1/\pi_i$ also makes the resulting estimator consistent, but it is less efficient than $\hat{\theta}_w$ using w_i in (6).

Parameter estimation : GMM method

- Because $\mathbf{z}$ is a nonresponse instrumental variable, we may assume

$$P(\delta = 1 \mid \mathbf{x}, y) = \pi(\phi_0 + \phi_1 \mathbf{u} + \phi_2 y)$$

  for some (ϕ_0, ϕ_1, ϕ_2).

- Kott and Chang (2008) idea: Construct a set of estimating equations such as

$$\sum_{i=1}^{n} \left\{ \frac{\delta_i}{\pi(\phi_0 + \phi_1 \mathbf{u}_i + \phi_2 y_i)} - 1 \right\} (1, \mathbf{u}_i, \mathbf{z}_i) = 0$$

  that are unbiased to zero.

- May have overidentified situation: Use the generalized method of moments (GMM).

GMM method

Example 2

- Suppose that we are interested in estimating the parameters in the regression model

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + e_i \tag{7}$$

where $E(e_i \mid \mathbf{x}_i) = 0$.

- Assume that y_i is subject to missingness and assume that

$$P(\delta_i = 1 \mid x_{1i}, x_{i2}, y_i) = \frac{\exp(\phi_0 + \phi_1 x_{1i} + \phi_2 y_i)}{1 + \exp(\phi_0 + \phi_1 x_{1i} + \phi_2 y_i)}.$$

Thus, x_{2i} is the nonresponse instrument variable in this setup.

Example 2 (Cont'd)

- A consistent estimator of ϕ can be obtained by solving

$$\hat{U}_2(\phi) \equiv \sum_{i=1}^{n} \left\{ \frac{\delta}{\pi(\phi; x_{1i}, y_i)} - 1 \right\} (1, x_{1i}, x_{2i}) = (0, 0, 0). \qquad (8)$$

  Roughly speaking, the solution to (8) exists almost surely if $E\{\partial \hat{U}_2(\phi)/\partial \phi\}$ is of full rank in the neighborhood of the true value of ϕ. If x_2 is vector, then (8) is overidentified and the solution to (8) does not exist. In the case, the GMM algorithm can be used.

- Finding the solution to $\hat{U}_2(\phi) = 0$ can be obtained by finding the minimizer of $Q(\phi) = \hat{U}_2(\phi)' \hat{U}_2(\phi)$ or $Q_W(\phi) = \hat{U}_2(\phi)' W \hat{U}_2(\phi)$ where $W = \{V(\hat{U}_2)\}^{-1}$.

Example 2 (Cont'd)

- Once the solution $\hat{\phi}$ to (8) is obtained, then a consistent estimator of $\beta = (\beta_0, \beta_1, \beta_2)$ can be obtained by solving

$$\hat{U}_1(\beta, \hat{\phi}) \equiv \sum_{i=1}^{n} \frac{\delta_i}{\hat{\pi}_i} \{y_i - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i}\} (1, x_{1i}, x_{2i}) = (0, 0, 0) \tag{9}$$

for β.

Asymptotic Properties

- The asymptotic variance of $\hat{\beta}$ obtained from (9) with $\hat{\phi}$ computed from the GMM can be obtained by

$$V(\hat{\theta}) \cong \left(\Gamma_a' \Sigma_a^{-1} \Gamma_a\right)^{-1}$$

where

$$\begin{aligned}
\Gamma_a &= E\{\partial \hat{U}(\theta)/\partial\theta\} \\
\Sigma_a &= V(\hat{U}) \\
\hat{U} &= (\hat{U}_1', \hat{U}_2')'
\end{aligned}$$

and $\theta = (\beta, \phi)$.

§4. Exponential tilting model approach

Exponential tilting method

Motivation

- Parameter θ defined by $E\{U(\theta; X, Y)\} = 0$.
- We are interested in estimating θ from an expected estimating equation:

$$\sum_{i=1}^{n} [\delta_i U(\theta; \mathbf{x}_i, y_i) + (1 - \delta_i) E\{U(\theta; \mathbf{x}_i, Y) \mid \mathbf{x}_i, \delta_i = 0\}] = 0. \quad (10)$$

- The conditional expectation in (10) can be evaluated by using

$$f(y|\mathbf{x}, \delta = 0) = f(y|\mathbf{x}) \frac{P(\delta = 0|\mathbf{x}, y)}{E\{P(\delta = 0|\mathbf{x}, y)|\mathbf{x}\}} \quad (11)$$

which requires correct specification of $f(y \mid \mathbf{x}; \theta)$. Known to be sensitive to the choice of $f(y \mid \mathbf{x}; \theta)$.

Exponential tilting method

Idea

Instead of specifying a parametric model for $f(y \mid \mathbf{x})$, consider specifying a parametric model for $f(y \mid \mathbf{x}, \delta = 1)$, denoted by $f_1(y \mid \mathbf{x})$. In this case,

$$f_0 (y_i \mid \mathbf{x}_i) = f_1 (y_i \mid \mathbf{x}_i) \times \frac{O (\mathbf{x}_i, y_i)}{E \{O (\mathbf{x}_i, Y_i) \mid \mathbf{x}_i, \delta_i = 1\}}, \tag{12}$$

where $f_\delta (y_i \mid \mathbf{x}_i) = f (y_i \mid \mathbf{x}_i, \delta_i = \delta)$ and

$$O (\mathbf{x}_i, y_i) = \frac{Pr (\delta_i = 0 \mid \mathbf{x}_i, y_i)}{Pr (\delta_i = 1 \mid \mathbf{x}_i, y_i)} \tag{13}$$

is the conditional odds of nonresponse.

Remark

- If the response probability follows from a logistic regression model
$$\pi(\mathbf{x}_i, y_i) \equiv Pr\left(\delta_i = 1 \mid \mathbf{x}_i, y_i\right) = \frac{\exp\left\{g(\mathbf{x}_i) + \phi y_i\right\}}{1 + \exp\left\{g(\mathbf{x}_i) + \phi y_i\right\}}, \tag{14}$$
where $g(\mathbf{x})$ is completely unspecified, the expression (12) can be simplified to
$$f_0\left(y_i \mid \mathbf{x}_i\right) = f_1\left(y_i \mid \mathbf{x}_i\right) \times \frac{\exp\left(\gamma y_i\right)}{E\left\{\exp\left(\gamma Y\right) \mid \mathbf{x}_i, \delta_i = 1\right\}}, \tag{15}$$
where $\gamma = -\phi$ and $f_1\left(y \mid \mathbf{x}\right)$ is the conditional density of y given $\mathbf{x}$ and $\delta = 1$.
- Model (15) states that the density for the nonrespondents is an exponential tilting of the density for the respondents. The parameter γ is the tilting parameter that determines the amount of departure from the ignorability of the response mechanism. If $\gamma = 0$, the the response mechanism is ignorable and $f_0(y|\mathbf{x}) = f_1(y|\mathbf{x})$.

Estimation of tilting parameter

- Sverchkov (2008) considered direct maximization of the observed likelihood for ϕ: Given a parametric model for $f_1(y \mid \mathbf{x})$ and $\pi(\mathbf{x}, y; \phi)$, find $\hat{\phi}$ that maximizes

$$l_{obs}(\phi) = \sum_{i=1}^{n} \delta_i \log \pi(\mathbf{x}_i, y_i; \phi) + \sum_{i=1}^{n} (1-\delta_i) \log \int \{1-\pi(\mathbf{x}_i, y; \phi)\} \hat{f}_1(y \mid \mathbf{x}_i) dy.$$

- Riddles et al. (2015) proposed an alternative computational tool that avoids computing the above integration using an EM-type algorithm.
- Semiparametric extension (Morikawa et al., 2015): Use a nonparametric density for $f_1(y \mid \mathbf{x})$.

A Toy Example: Categorical Data (All dichotomous)

Example (SRS, $n = 10$)

ID	Weight	x_1	x_2	y
1	0.1	1	0	1
2	0.1	1	1	1
3	0.1	0	1	M
4	0.1	1	0	0
5	0.1	0	1	1
6	0.1	1	0	M
7	0.1	0	1	M
8	0.1	1	0	0
9	0.1	0	0	0
10	0.1	1	1	0

M: Missing

A Toy Example (Cont'd)

Assume $P(\delta = 1 \mid x_1, x_2, y) = \pi(x_1, y)$

ID	Weight	x_1	x_2	y
1	0.1	1	0	1
2	0.1	1	1	1
3	$0.1 \cdot w_{3,0}$	0	1	0
	$0.1 \cdot w_{3,1}$	0	1	1
4	0.1	1	0	0
5	0.1	0	1	1

$$\begin{aligned} w_{3,j} &= \hat{P}(Y = j \mid X_1 = 0, X_2 = 1, \delta = 0) \\ &\propto \hat{P}(Y = j \mid X_1 = 0, X_2 = 1, \delta = 1)\frac{1 - \hat{\pi}(0, j)}{\hat{\pi}(0, j)} \end{aligned}$$

with

$$w_{3,0} + w_{3,1} = 1$$

A Toy Example (Cont'd)

ID	Weight	x_1	x_2	y
6	$0.1 \cdot w_{6,0}$	1	0	0
	$0.1 \cdot w_{6,1}$	1	0	1
7	$0.1 \cdot w_{7,0}$	0	1	0
	$0.1 \cdot w_{7,1}$	0	1	1
8	0.1	1	0	0
9	0.1	0	0	0
10	0.1	1	1	0

$$
\begin{aligned}
w_{6,j} &\propto \hat{P}(Y = j \mid X_1 = 1, X_2 = 0, \delta = 1)\frac{1 - \hat{\pi}(1, j)}{\hat{\pi}(1, j)} \\
w_{7,j} &\propto \hat{P}(Y = j \mid X_1 = 0, X_2 = 1, \delta = 1)\frac{1 - \hat{\pi}(0, j)}{\hat{\pi}(0, j)}
\end{aligned}
$$

with

$$w_{6,0} + w_{6,1} = w_{7,0} + w_{7,1} = 1.$$

Example (Cont'd)

- E-step: Compute the conditional probability using the estimated response probability $\hat{\pi}_{ab}$.
- M-step: Update the response probability using the fractional weights. For fully nonparametric model,

$$\hat{\pi}_{ab} = \frac{\sum_{\delta_i=1} I(x_{1i} = a, y_i = b)}{\sum_{\delta_i=1} I(x_{1i} = a, y_i = b) + \sum_{\delta_i=0} \sum_{j=0}^{1} w_{i,j} I(x_{1i} = a, y_{ij}^* = b)}$$

- The solution from the proposed method is $\hat{\pi}_{11} = 1$, $\hat{\pi}_{10} = 3/4$, $\hat{\pi}_{01} = 1/3$, $\hat{\pi}_{00} = 1$.

A Toy Example (Cont'd)

Example (Cont'd)

- The method can be viewed as a fractional imputation method of Kim (2011).
- On the other hand, GMM method is more close to nonresponse weighting adjustment.

Example GMM method

ID	Wgt 1	Wgt2	x_1	x_2	y
1	0.1	$0.1\hat{\pi}_{11}^{-1}$	1	0	1
2	0.1	$0.1\hat{\pi}_{11}^{-1}$	1	1	1
3	0.1	0.0	0	1	M
4	0.1	$0.1\hat{\pi}_{10}^{-1}$	1	0	0
5	0.1	$0.1\hat{\pi}_{01}^{-1}$	0	1	1
6	0.1	0.0	1	0	M
7	0.1	0.0	0	1	M
8	0.1	$0.1\hat{\pi}_{10}^{-1}$	1	0	0
9	0.1	$0.1\hat{\pi}_{00}^{-1}$	0	0	0
10	0.1	$0.1\hat{\pi}_{10}^{-1}$	1	1	0

M: Missing

A Toy Example (Cont'd)

- GMM method: Calibration equation

$$\sum_i \frac{\delta_i}{\hat{\pi}_i} I(x_{1i} = a, x_{2i} = b) = \sum_i I(x_{1i} = a, x_{2i} = b).$$

1. $X_1 = 1, X_2 = 1$: $\hat{\pi}_{11}^{-1} + \hat{\pi}_{10}^{-1} = 2$
2. $X_1 = 1, X_2 = 0$: $\hat{\pi}_{11}^{-1} + \hat{\pi}_{10}^{-1} + \hat{\pi}_{10}^{-1} = 4$
3. $X_1 = 0, X_2 = 1$: $\hat{\pi}_{01}^{-1} = 3$
4. $X_1 = 0, X_2 = 0$: $\hat{\pi}_{00}^{-1} = 1$.

- The solution of GMM method does not exist.

§5 Callbacks

Basic Setup

- Consider a non-ignorable response mechanism of the form

$$Pr\left(\delta_i = 1 \mid \mathbf{x}_i, y_i\right) = \pi(\phi; \mathbf{x}_i, y_i) = \frac{\exp(\phi_0 + \mathbf{x}_i\phi_1 + y_i\phi_2)}{1 + \exp(\phi_0 + \mathbf{x}_i\phi_1 + y_i\phi_2)}. \quad (16)$$

- Clearly, the score equation cannot be solved because y_i are not observed when $\delta_i = 0$.
- To estimate the parameters in (16), we consider the special case when there are some callbacks among nonrespondents. That is, among the elements with $\delta_i = 0$, further efforts are made to obtain the observation of y_i. Let $\delta_{2i} = 1$ if the element i is selected for a callback or $\delta_i = 1$ and $\delta_{2i} = 0$ otherwise. We assume that the selection mechanism for the callback depends only δ_i. That is,

$$Pr\left(\delta_2 = 1 \mid \mathbf{x}, y, \delta\right) = \begin{cases} 1 & \text{if } \delta = 1 \\ \nu & \text{if } \delta = 0 \end{cases} \quad (17)$$

for some $\nu \in (0, 1]$.

Lemma 6.2

Lemma

Assume that the response mechanism satisfies (16) and the followup sample is randomly selected among the nonrespondents with probability ν. Then, the response probability among the set with $\delta_{i2} = 1$ can be expressed as

$$Pr\left(\delta_i = 1 \mid \mathbf{x}_i, y_i, \delta_{2i} = 1\right) = \frac{\exp(\phi_0^* + \mathbf{x}_i\phi_1^* + y_i\phi_2^*)}{1 + \exp(\phi_0^* + \mathbf{x}_i\phi_1^* + y_i\phi_2^*)} \tag{18}$$

where $\phi_0^* = \phi_0 - \ln(\nu)$, $(\phi_1^*, \phi_2^*) = (\phi_1, \phi_2)$, and (ϕ_0, ϕ_1, ϕ_2) is defined in (16).

Proof of Lemma 6.2

By Bayes formula,

$$\frac{Pr\left(\delta = 1 \mid \mathbf{x}, y, \delta_2 = 1\right)}{Pr\left(\delta = 0 \mid \mathbf{x}, y, \delta_2 = 1\right)} = \frac{Pr\left(\delta_2 = 1 \mid \mathbf{x}, y, \delta = 1\right)}{Pr\left(\delta_2 = 1 \mid \mathbf{x}, y, \delta = 0\right)} \times \frac{Pr\left(\delta = 1 \mid \mathbf{x}, y\right)}{Pr\left(\delta = 0 \mid \mathbf{x}, y\right)}.$$

By (17), the above formula reduces to

$$\frac{Pr\left(\delta = 1 \mid \mathbf{x}, y, \delta_2 = 1\right)}{Pr\left(\delta = 0 \mid \mathbf{x}, y, \delta_2 = 1\right)} = \frac{1}{\nu} \times \frac{Pr\left(\delta = 1 \mid \mathbf{x}, y\right)}{Pr\left(\delta = 0 \mid \mathbf{x}, y\right)}.$$

Taking the logarithm of the above equality, we have

$$\phi_0^* + \phi_1^* \mathbf{x} + \phi_2^* y = \phi_0 - \ln(\nu) + \phi_1 \mathbf{x} + \phi_2 y.$$

Because the above relationship holds for all $\mathbf{x}$ and y, we have $\phi_0^* = \phi_0 - \ln(\nu)$ and $(\phi_1^*, \phi_2^*) = (\phi_1, \phi_2)$.

Remark

- By Lemma 6.2, the MLE of ϕ^* can be obtained by maximizing the conditional likelihood. That is, we solve

$$\sum_{i=1}^{n} \delta_{2i} \left\{ \delta_i - \pi(\phi^*; \mathbf{x}_i, y_i) \right\} (\mathbf{x}_i, y_i) = 0 \tag{19}$$

  and then applying the transformation in Lemma 6.2. In particular, the MLE for the slope (ϕ_1, ϕ_2) in (16) can be directly computed by solving (19).
- Variance-covariance matrix of $(\hat{\phi}_1, \hat{\phi}_2)$ is the same as that of $(\hat{\phi}_1^*, \hat{\phi}_2^*)$.

Alternative method

- Let $f_\delta(x, y)$ be the joint density of (x, y) given δ, $\delta = 0, 1$. The response probability can be computed by, using Bayes formula,

$$P(\delta = 1 \mid x, y) = \frac{\pi f_1(x, y)}{\pi f_1(x, y) + (1 - \pi) f_0(x, y)},$$

  where $\pi = P(\delta = 1)$. We can use the initial respondents to estimate $f_1(x, y)$ and use the follow-up data to estimate $f_0(x, y)$.
- If we fit $f_1(x, y)$ and $f_0(x, y)$ as normal distributions (with the same variance-covariance matrix), the response probability follows from a logistic regression model.

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